The effect of viscosity near the cylindrical boundaries of a rotating fluid with a horizontal temperature gradient

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Under certain conditions, the motion caused in an annulus of fluid by rotating it about its (vertical) axis of symmetry and at the same time subjecting it to a radial temperature gradient has been shown by Hide (1958) to be mostly concentrated in a narrow jet stream which meanders between the inner and outer cylindrical boundaries of the fluid in a regular wave pattern: this wave pattern has a small angular velocity relative to the cylindrical walls containing the fluid. A theoretical solution has been found by Davies (1959) which is valid in the main body of the fluid: this solution neglects viscosity (which is permissible except near the boundaries of the fluid), and is related to the absolute angular velocity of the wave pattern. The present paper introduces viscous boundary layers between the main body of the fluid and the cylindrical walls, in an attempt to find a relation between the angular velocity of the wave pattern and that of the walls. That this is only partially successful is due to the presence of the boundary layer at the rigid surface at the bottom of the fluid (which is rotating with the same angular velocity as the cylindrical walls): this layer is ignored in the present theory. In addition to this contribution towards a complete explanation of the steady motion, the theory describes qualitatively certain periodic oscillations (vacillation) which were observed by Hide in his experiments.

1. Introduction

The phenomena which are observed when an annular cylinder of fluid is rotated about its (vertical) axis of symmetry in the presence of a radial temperature gradient have been described in detail by Hide (1958). He finds that, for sufficiently small values of the parameter

$$\Theta = \frac{gh}{(b-a)^2\,\Omega^2}\frac{\Delta\rho}{\rho_0},$$

where g is the acceleration due to gravity, a, b, h are the dimensions of the apparatus as shown in figure 1, Ω is the angular velocity of the apparatus, $\Delta\rho$ is the difference in densities of the working fluid at the two cylindrical boundaries and ρ_0 is the mean density, a motion develops which is mainly horizontal and which consists of a wave-pattern in which a clearly defined jet-stream can be observed. A typical surface pattern, consisting of three lobes, is shown in figure 2: the motion relative to the rotating cylinders is almost entirely in a

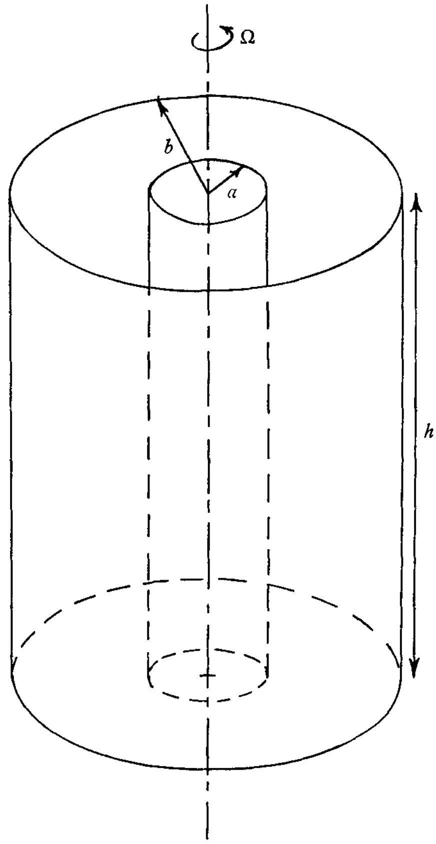

FIGURE 1. The dimensions of the apparatus.

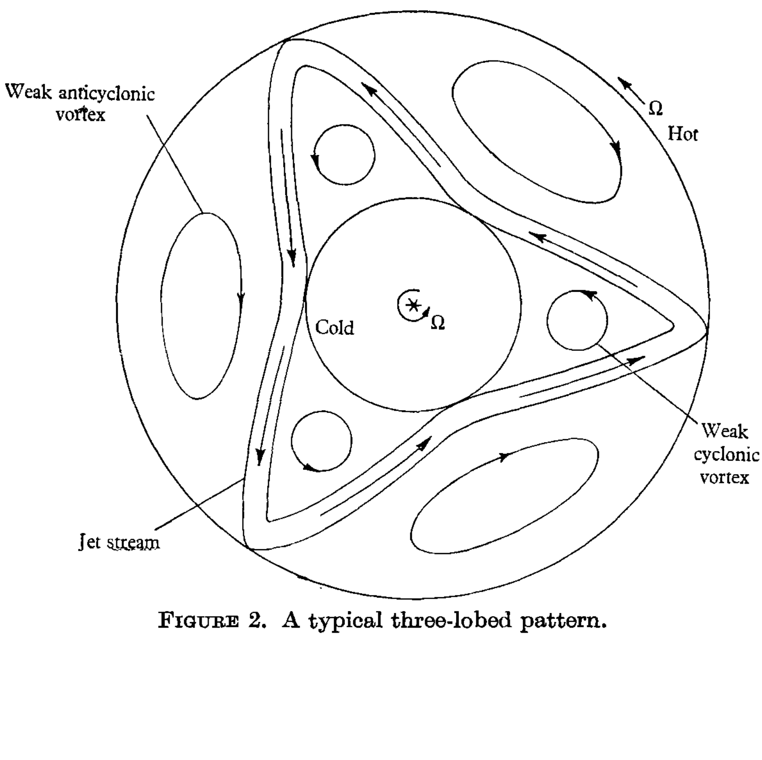

FIGURE 2. A typical three-lobed pattern.

narrow jet-stream, although weak vortices do exist in the positions shown. The wave pattern itself rotates slowly relative to the containing cylinders, with absolute angular velocity ω†, such that

$$\frac{\omega-\Omega}{\Omega} = 5{\cdot}82\times 10^{-2}\frac{gh}{(b^2-a^2)\,\Omega^2}\frac{\Delta\rho}{\rho_0}, \tag{1}$$

where $\Delta\rho$ is taken to be positive when the outer cylinder is hotter than the inner cylinder. He finds also that, as Θ increases, the number, m, of lobes in the wave pattern increases from a minimum value $m_{\min}$ to a maximum value $m_{\max}$, both of these quantities depending on the geometry of the apparatus. When Θ is increased still further, there occurs, in certain cases, a phenomenon which he calls 'vacillation': here there is a periodic break-up of the pattern, as described below. First of all, the pattern develops a slight backward tilt at the outer boundary; this then disappears, and a forward tilt develops which increases in magnitude until the jet-stream rolls up on itself, and a pattern of strong cyclonic vortices is formed. This, in turn, breaks down, and a new pattern similar to that of figure 2 is formed, and the whole cycle is repeated. (There is some doubt as to the length of time such a motion will persist if there is no external disturbance given to the fluid; since, however, the motion is observed to persist for several hundred revolutions of the apparatus after a single disturbance such as an accidental knock of the turntable supporting it, it is permissible to regard the phenomenon as persisting for very long periods of time.) In a typical example, Hide finds that for a three-lobe pattern, where $a = 1{\cdot}06$ cm, $b = 4{\cdot}85$ cm, $h = 10$ cm, $\Delta\rho/\rho_0 = 11{\cdot}8\times 10^{-3}$, $\Omega = 4{\cdot}71$ rad./sec, a vacillation cycle develops whose period is 28 sec.

Theoretical studies of the steady motion described above have been made by Rogers (1959) and Davies (1959). They find that, for small values of the parameter Θ, it is permissible to neglect inertia and viscous effects in the main body of the fluid (so that the motion is essentially given by a balance between Coriolis forces and buoyancy forces), but that heat transfer is effected both by conduction and by convection. Using these approximations, Davies (1959) finds a solution for a pattern with m lobes which has a temperature distribution of the form

$$T = \{F(R)+\phi(R)\cos[m\theta-\Psi(R)]\}\,(2\rho_0\,\omega\kappa/\alpha_0\,gh) + \text{a linear function of } z,$$

a pressure distribution of the form

$$p = (2\omega\rho_0/m)\,\{f(R)+\phi(R)\sin[m\theta-\Psi(R)] + (\beta_0+z)\,\phi(R)\cos[m\theta-\Psi(R)]\} + \text{a quadratic function of } z,$$

and a transverse velocity distribution of the form

$$v = \frac{\kappa}{mr}\left\{\frac{df}{dR}+\left[\frac{d\phi}{dR}+\phi\frac{d\Psi}{dR}(\beta_0+z)\right]\sin[m\theta-\Psi(R)] + \left[\frac{d\phi}{dR}(\beta_0+z)-\phi\frac{d\Psi}{dR}\right]\cos[m\phi-\Psi(R)]\right\}.$$

In these expressions (r,θ,z) are cylindrical-polar co-ordinates rotating with the fluid ($z = 0$ being the base of the apparatus); $R = \log_e(r/b)$; α_0 is the coefficient of

† N.B. Hide takes ω as the *relative* angular velocity.

thermal expansion and κ the thermometric conductivity of the fluid; β_0 is an arbitrary constant; and F, f, ϕ and Ψ' are functions of R which can be determined once Ψ' is known. As a particular example he takes

$$\frac{d\Psi'}{dR} = \gamma_0 \phi;$$

then γ_0 is a constant which is a measure of the transfer of westerly angular momentum (that is, the angular momentum in the same sense as that of the apparatus) from the outer to the inner boundary. Then he finds that

$$F(R) = jR - \tfrac{1}{2}\phi_1^2 \int_{-R}^{0} \operatorname{sn}^2\{-\tfrac{1}{2}R\phi_2(1+2\gamma_0^2)^{\frac{1}{2}}, k\}\, dR,$$

$$f(R) = (\beta_0 + z)\left\{jR - \tfrac{1}{2}\phi_1^2 \int_{-R}^{0} \operatorname{sn}^2\{-\tfrac{1}{2}R\phi_2(1+2\gamma_0^2)^{\frac{1}{2}}, k\}\, dR\right\} - 3\gamma_0\phi,$$

$$\phi(R) = \phi_1 \operatorname{sn}\{-\tfrac{1}{2}R\phi_2(1+2\gamma_0^2)^{\frac{1}{2}}, k\},$$

where $$\phi_1 = \frac{4K}{R_0(1+2\gamma_0^2)^{\frac{1}{2}}}, \quad \phi_2 = \frac{4Kk}{R_0(1+2\gamma_0^2)^{\frac{1}{2}}}, \quad j = m^2 + \frac{4K^2(1+k^2)}{R_0^2},$$

$$1/\epsilon_0 = -gh\Delta\rho/2\omega\rho_0\kappa = mR_0 + \nu_0^2/mR_0, \quad R_0 = \log_e(b/a),$$

$$\nu_0^2 = \{4K/(1+2\gamma_0^2)\}\{2E - K[(1-k^2) - 2\gamma_0^2(1+k^2)]\},$$

$$K = \int_0^{\frac{1}{2}\pi} \frac{du}{(1-k^2\sin^2 u)^{\frac{1}{2}}}, \quad E = \int_0^{\frac{1}{2}\pi} (1-k^2\sin^2 u)^{\frac{1}{2}}\, du.$$

For m less than its maximum value, these equations are sufficient, when a, b, h, $\Delta\rho/\rho_0$ and ω are given, to determine first m, then k and then the functions F, f, ϕ in terms of β_0 and γ_0. The first insufficiency of the theory is immediately apparent, since ω is not known: one of the objects of the present paper is to find a theoretical expression for the ratio $(\omega - \Omega)/\Omega$ so that, if Ω is known, so is ω. It will be found, in fact, that this ratio is a function of γ_0, and so is still insufficient to complete the theory: the *form* of the ratio, however, is found to be that of equation (1), which lends support to the theory, and which can therefore be used to obtain an empirical value for γ_0. A relation between the two constants β_0 and γ_0 is also obtained.

The method used to obtain these results is to try to insert viscous boundary layers between the main body of fluid in the motion found by Davies (which will be referred to as *the inviscid solution* throughout this paper) and the two cylindrical walls. That such layers must exist follows from the fact that in the inviscid solution, slipping occurs at one or both cylindrical boundaries. If it is assumed that the motion is everywhere steady, it is possible to obtain two relations between the constants β_0, γ_0 and $(\omega - \Omega)/\Omega$, as already mentioned.

It is possible to extend the idea a little further by assuming a periodic variation with time in the viscous boundary layer. Since any disturbance to the fluid which may result in vacillation is likely to emanate from the walls, it seems reasonable to expect such a periodic variation with time to exist, and to have a period comparable with that of the vacillation cycle. A quantitative solution is not found, but a qualitative picture emerges of a forward tilting of the wave-pattern at the outer boundary which travels in the forward direction of motion of the fluid—

that is, a periodic rolling-up of the jet-stream is predicted, as occurs in one phase of the vacillation cycle. The slight backward tilting in another phase of the cycle is shown to be always present in the inviscid solution.

The boundary layers at the base of the fluid and at the free surface are neglected in this paper; of these, it is only the former which is likely to be of any significance. The main flow, however, is not in this problem *forced* through the viscous boundary layers: instead the flow is superimposed upon a 'solid rotation' by buoyancy forces, so that the effect of the viscous boundary layers is that of a drag. Since, in the main flow, the fluid velocity relative to the apparatus increases linearly with height, that near the base is comparatively small: we can therefore expect the drag from the base boundary layer to be much smaller than that from the cylindrical walls. It will be found, however, that the solution of the equations for the boundary layers on the cylindrical walls cannot be found uniquely unless one boundary condition is known at the base of these walls: as will be shown in more detail in § 2, it is reasonable as a first approximation to ignore the base boundary layer on the average, and to take the velocity of the fluid relative to the apparatus to be zero at the base. Because of the neglect of these effects, and hence of some of the highest-order derivatives in the equations of motion, it is not surprising that some arbitrary constants remain in the solution. The next step in the investigation of the flow must, of course, be to include the boundary layer on the base.

2. Simplification of the equations of motion. Boundary conditions

Using cylindrical-polar co-ordinates (r, θ, z) which rotate with the wave pattern (i.e. with absolute angular velocity ω), the equations of horizontal motion, continuity and heat transfer for a fluid are respectively

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}-2\omega v = -\frac{1}{\rho_0}\frac{\partial p}{\partial r}$$

$$+\nu\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}-\frac{u}{r^2}-\frac{2}{r^2}\frac{\partial v}{\partial \theta}\right],$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+2\omega u = -\frac{1}{\rho_0 r}\frac{\partial p}{\partial \theta}$$

$$+\nu\left[\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{\partial^2 v}{\partial z^2}+\frac{2}{r^2}\frac{\partial u}{\partial \theta}-\frac{v+\omega r}{r^2}\right],$$

$$0 = -g-\frac{1}{\rho}\frac{\partial p}{\partial z},$$

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \theta} = 0,$$

and

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial r}+\frac{v}{r}\frac{\partial T}{\partial \theta} = \kappa\left[\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}\right],$$

where ν is the kinematic viscosity, u is the radial component of the velocity of the fluid and v is its transverse component relative to the rotating axes. If we now introduce non-dimensional variables by means of the substitutions

$$v = \omega a(\Delta\rho/\rho_0)\,\bar{v}, \quad u = \mathscr{R}^{-\frac{1}{2}}\omega a(\Delta\rho/\rho_0)\,\bar{u},$$
$$r = a(1+\mathscr{R}^{-\frac{1}{2}}\bar{r}), \quad t = \bar{t}/\omega, \quad z = h\bar{z},$$

where $\mathscr{R} = \omega a^2/\nu$ is a Reynolds number, and then neglect terms of order $\mathscr{R}^{-\frac{1}{2}}$ or $\Delta\rho/\rho_0$ compared with unity, and also a term of order $\mathscr{R}^{-1}$ compared with a term of order $\Delta\rho/\rho_0$ (which is permissible near a boundary unless $\Delta\rho/\rho_0$ is very small indeed), the first two of these equations reduce to

$$2\rho_0\omega v = \partial p/\partial r, \tag{2}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0 a}\frac{\partial p}{\partial \theta} + \nu\frac{\partial^2 v}{\partial r^2}, \tag{3}$$

and the last two reduce to

$$\frac{\partial u}{\partial r} + \frac{1}{a}\frac{\partial v}{\partial \theta} = 0, \tag{4}$$

$$\frac{\partial T}{\partial t} = \kappa\frac{\partial^2 T}{\partial r^2}.$$

These approximations are a combination of those used in the usual boundary-layer theory, and those used in flows having a small Rossby number. The resulting equations imply:

(i) The transverse velocity (imposed from outside—that is, from the main body of the fluid) causes a Coriolis force which requires a pressure field to balance it.

(ii) The transverse variation of the pressure field is balanced by diffusion of vorticity, but will also be explicitly affected if there is any unsteadiness in the motion.

(iii) The radial velocity component is due entirely to considerations of continuity.

(iv) The temperature field in the region concerned is independent of the motion in the region: since, in the inviscid solution, the temperature boundary condition was properly satisfied, it is quite sufficient to take T = constant throughout the region considered here.

Substituting now from (2) into (3), we find that p satisfies the linear differential equation

$$\frac{\partial^2 p}{\partial t\,\partial r} = -\frac{2\omega}{a}\frac{\partial p}{\partial \theta} + \nu\frac{\partial^3 p}{\partial r^3}, \tag{5a}$$

and this is valid in the neighbourhood of the inner cylinder, $r = a$. In the neighbourhood of the outer cylinder, $r = b$, a must be replaced everywhere by b, and it is convenient to indicate the values of the dependent variables in this region by an asterisk. Hence p^* satisfies the equation

$$\frac{\partial^2 p^*}{\partial t\,\partial r} = -\frac{2\omega}{b}\frac{\partial p^*}{\partial \theta} + \nu\frac{\partial^3 p^*}{\partial r^3}. \tag{5b}$$

Once p is known, v can be found from (2) and u from (4); similarly v^* and u^* can be found from the corresponding equations with a replaced by b when p^* is known.

The boundary conditions in the neighbourhood of the inner cylinder are as follows.

(i) When $r = a$, $u = 0$ and $v = -a(\omega - \Omega)$ for all θ and t. (6*a*)

This is the condition of no slipping at the inner cylinder.

(ii) When $r - a \to \infty$, $v \to v_0(z) + v_1(z)\cos m\theta + v_2(z)\sin m\theta$, (7*a*)

where v_0, v_1, v_2 are given by Davies (1959), and are of the form

$$v_0 + v_1\cos m\theta + v_2\sin m\theta = (\kappa/ma)\{(\beta_0 + z/h)j - \tfrac{3}{2}\gamma_0(1+2\gamma_0^2)^{\frac{1}{2}}\phi_1\phi_2 + \tfrac{1}{2}(1+2\gamma_0^2)^{\frac{1}{2}}\phi_1\phi_2[\sin(m\theta - \Psi_0) + (\beta_0 + z/h)\cos(m\theta - \Psi_0)]\}, \quad (8a)$$

using the notation already defined in the introduction to this paper, and writing Ψ_0 for $(\Psi)_{R=-R_0}$. This is the matching condition between the boundary layer and the inviscid flow.

(iii) When $z = 0$, $v = -a(\omega - \Omega) + v_1(z)\cos m\theta + v_2(z)\sin m\theta$, (9*a*)

where $v_1(z)$ and $v_2(z)$ are the functions defined by equation (8*a*). This condition is equivalent to saying that the part of the azimuthal velocity relative to the apparatus which is independent of θ is zero outside the base boundary layer. From the known form of the external flow this relative velocity is certainly much smaller than that near the surface (since it increases linearly with z), and is also observed to be comparatively small in the experimental work. Further, if there exists a difference between the mean flow velocity and the velocity of the base, it seems likely that the base boundary layer would increase in thickness indefinitely, until it engulfed the flow. That this does not happen in many rotating flows already known (e.g. Stewartson 1957) is due to the fact that the boundary under consideration is that which originally causes the motion; in the present problem, the main motion is set up by another mechanism (the interaction of buoyancy and Coriolis force which are both body forces), so that the boundary acts only as a drag.

The corresponding conditions in the neighbourhood of the outer cylinder are:

(i) when $r = b$, $u^* = 0$ and $v^* = -b(\omega - \Omega)$ for all θ and t; (6*b*)

(ii) when $r - b \to -\infty$, $v \to v_0^*(z) + v_1^*(z)\cos m\theta + v_2^*(z)\sin m\theta$, (7*b*)

where
$$v_0^* + v_1^*\cos m\theta + v_2^*\sin m\theta = (\kappa/mb)\{(\beta_0 + z/h)j + \tfrac{3}{2}\gamma_0(1+2\gamma_0^2)^{\frac{1}{2}}\phi_1\phi_2 - \tfrac{1}{2}(1+2\gamma_0^2)^{\frac{1}{2}}\phi_1\phi_2[\sin(m\theta - \Psi_0^*) + (\beta_0 + z/h)\cos(m\theta - \Psi_0^*)]\}, \quad (8b)$$

and
$$\Psi_0^* = (\Psi)_{R=0} = \Psi_0 - \frac{2\gamma_0}{(1+2\gamma_0^2)^{\frac{1}{2}}}\log_e\left(\frac{1-k}{1+k}\right);$$

(iii) when $z = 0$, $v^* = -b(\omega - \Omega) + v_1^*(z)\cos m\theta + v_2^*(z)\sin m\theta$, (9*b*)

where $v_1^*(z)$ and $v_2^*(z)$ are the functions defined by (8*b*).

The problem is therefore reduced to that of solving equations (5), subject to the boundary conditions (6), (7) and (9). The solution will be discussed in the following sections.

3. The steady solution

When all the variables are independent of time, equation ($5a$) reduces to

$$-\frac{2\omega}{a}\frac{\partial p}{\partial \theta}+\nu\frac{\partial^3 p}{\partial r^3} = 0. \tag{10}$$

Since the solution is periodic in θ, with wave-number m, we can expect a solution of the form

$$p(r,\theta,z) = p_0(r,z)+p_1(r,z)\,e^{im\theta}+p_2(r,z)\,e^{-im\theta}. \tag{11}$$

This may be substituted into equation (10), and then

$$\partial^3 p_0/\partial r^3 = 0 \quad \text{giving} \quad p_0 = 2\omega\rho_0[A_0(z)+B_0(z)\,(r-a)+\tfrac{1}{2}C_0(z)\,(r-a)^2],$$

and

$$(2i\omega m/a)\,p_1 = \nu\partial^3 p_1/\partial r^3, \quad -(2i\omega m/a)\,p_2 = \nu\partial^3 p_2/\partial r^3.$$

These may be written

$$\partial^3 p_1/\partial r^3 = \mu^3 i p_1, \quad \partial^3 p_2/\partial r^3 = -\mu^3 i p_2, \quad \text{where} \quad \mu = (2\omega m/\nu a)^{\frac{1}{3}}.$$

Now the equation $\partial^3 p_1/\partial r^3 = \mu^3 i p_1$ has solutions of the form $e^{-i\mu r}$, $e^{(\sqrt{3}+i)\,\mu r/2}$ and $e^{(-\sqrt{3}+i)\,\mu r/2}$. The expression (11), after some manipulation, can be shown to be of the form

$$\begin{aligned} p = 2\omega\rho_0[A_0+B_0(r-a)+\tfrac{1}{2}C_0(r-a)^2+A_1\cos[m\theta-\mu(r-a)]+A_2\sin[m\theta-\mu(r-a)] \\ +e^{\sqrt{3}\,\mu(r-a)/2}\{B_1\cos[m\theta+\tfrac{1}{2}\mu(r-a)]+B_2\sin[m\theta+\tfrac{1}{2}\mu(r-a)]\} \\ +e^{-\sqrt{3}\,\mu(r-a)/2}\{C_1\cos[m\theta+\tfrac{1}{2}\mu(r-a)]+C_2\sin[m\theta+\tfrac{1}{2}\mu(r-a)]\}], \end{aligned}$$

where $A_0, A_1, A_2, B_0, B_1, B_2, C_0, C_1, C_2$ are all arbitrary functions of z. Since all the boundary conditions are on the transverse velocity, v, it is more convenient to differentiate this with respect to r, and use (2) to obtain

$$\begin{aligned} v = B_0+C_0(r-a)+F\cos[m\theta-\mu(r-a)+\alpha] \\ +e^{\sqrt{3}\,\mu(r-a)/2}G\cos[m\theta+\tfrac{1}{2}\mu(r-a)+\beta] \\ +e^{-\sqrt{3}\,\mu(r-a)/2}H\cos[m\theta+\tfrac{1}{2}\mu(r-a)+\gamma], \end{aligned} \tag{12a}$$

where F, G, H, α, β, γ are new arbitrary functions of z which are related to $A_1, A_2, B_1, B_2, C_1, C_2$. Similarly, in the neighbourhood of the outer cylinder, the transverse velocity is given by

$$\begin{aligned} v^* = B_0^*+C_0^*(r-b)=F^*\cos[m\theta-\mu^*(r-b)+\alpha^*] \\ +e^{\sqrt{3}\,\mu^*(r-b)/2}G^*\cos[m\theta+\tfrac{1}{2}\mu^*(r-b)+\beta^*] \\ +e^{-\sqrt{3}\,\mu^*(r-b)/2}H^*\cos[m\theta+\tfrac{1}{2}\mu^*(r-b)+\gamma^*], \end{aligned} \tag{12b}$$

where $F^*, G^*, H^*, \alpha^*, \beta^*, \gamma^*$ are arbitrary functions of z. In the two equations (12), the constants μ and μ^* are defined by

$$\mu = (2\omega m/\nu a)^{\frac{1}{3}}, \quad \mu^* = (2\omega m/\nu b)^{\frac{1}{3}}. \tag{13}$$

It is now possible to apply the boundary conditions to the solutions (12). First of all, using ($6a$) and ($6b$), we find that

$$B_0 = -a(\omega-\Omega), \quad B_0^* = -b(\omega-\Omega),$$

$$F\cos\alpha+G\cos\beta+H\cos\gamma = 0, \quad F\sin\alpha+G\sin\beta+H\sin\gamma = 0,$$

$$F^*\cos\alpha^*+G^*\cos\beta^*+H^*\cos\gamma^* = 0, \quad F^*\sin\alpha^*+G^*\sin\beta^*+H^*\sin\gamma^* = 0.$$

Applying the boundary conditions (7*a*) and (7*b*), it is apparent, first of all, that

$$G = 0, \quad H^* = 0.$$

The foregoing equations then give

$$\alpha = \gamma, \quad \alpha^* = \beta^*, \quad F+H = 0, \quad F^*+G^* = 0.$$

At this stage, the equations (12*a*) and (12*b*) have been reduced to

$$\begin{aligned} v &= -a(\omega-\Omega)+C_0(r-a)+F\cos[m\theta-\mu(r-a)+\alpha] \\ &\quad -F\,e^{-\sqrt{3}\mu(r-a)/2}\cos[m\theta+\tfrac{1}{2}\mu(r-a)+\alpha], \\ v^* &= -b(\omega-\Omega)+C_0^*(r-b)+F^*\cos[m\theta-\mu^*(r-b)+\alpha^*] \\ &\quad -F^*\,e^{\sqrt{3}\mu^*(r-b)/2}\cos[m\theta+\tfrac{1}{2}\mu^*(r-b)+\alpha^*]. \end{aligned}$$

Comparing this with the forms (7*a*) and (7*b*), it is immediately apparent that it is convenient to take a boundary layer of finite thickness: a suitable position to choose as the edge of the boundary layer is where $e^{-\sqrt{3}\mu(r-a)/2}$ (or $e^{\sqrt{3}\mu^*(r-b)/2}$) is negligible. This quantity is approximately 0·005 when $r-a = \delta$ (or $b-r = \delta^*$) if

$$\mu\delta = \mu^*\delta^* = 2\pi. \tag{14}$$

This is taken to give suitable values for δ and δ^*. Then, when $r-a = \delta$, the boundary condition (7*a*) gives

$$v_0+v_1\cos m\theta+v_2\sin m\theta = -a(\omega-\Omega)+(2\pi/\mu)\,C_0+F\cos(m\theta+\alpha), \tag{15a}$$

and when $b-r = \delta^*$, the boundary condition (7*b*) gives

$$v_0^*+v_1^*\cos m\theta+v_2^*\sin m\theta = -b(\omega-\Omega)-(2\pi/\mu^*)\,C_0^*+F^*\cos(m\theta+\alpha^*). \tag{15b}$$

It follows at once that

$$F\cos\alpha = v_1, \quad F\sin\alpha = v_2, \quad F^*\cos\alpha^* = v_1^*, \quad F^*\sin\alpha^* = v_2^*, \tag{16}$$

where v_1, v_2, v_1^*, v_2^* are given in equations (8*a*) and (8*b*). The relations (16) determine the constants F, α, F^*, α^* completely, and the only arbitrariness left in the solution is in the constants C_0 and C_0^*. Still using the equations (15*a*) and (15*b*) we have

$$-a(\omega-\Omega)+(2\pi/\mu)\,C_0 = v_0 = (\kappa/ma)\,[(\beta_0+z/h)\,j-\tfrac{3}{2}\gamma_0(1+2\gamma_0^2)^{\frac{1}{2}}\,\phi_1\,\phi_2], \tag{17a}$$

and

$$-b(\omega-\Omega)-(2\pi/\mu^*)\,C_0^* = v_0^* = (\kappa/mb)\,[(\beta_0+z/h)\,j+\tfrac{3}{2}\gamma_0(1+2\gamma_0^2)^{\frac{1}{2}}\,\phi_1\,\phi_2]. \tag{17b}$$

Hence

$$(2\pi/\mu)\,C_0 = \bar{C}_0+(\kappa j/mha)\,z, \quad -(2\pi/\mu^*)\,C_0^* = \bar{C}_0^*+(\kappa j/mhb)\,z,$$

where $\bar{C}_0$ and $\bar{C}_0^*$ are constants.

Applying the conditions (9*a*) and (9*b*), we see that $\bar{C}_0$ and $\bar{C}_0^*$ are both zero; as stated before, however, the condition cannot be satisfied completely, as this would imply that $v_1 = v_2 = v_1^* = v_2^* = 0$ when $z = 0$, and this is not in general true. It will be necessary, therefore, to regard this as an approximate condition and to satisfy it only on the average with respect to θ. When a solution has been found for the boundary layer near $z = 0$, this can be used to replace the conditions (9), and hence to give a better picture. Then the equations (17a) and (17b) yield the relations

$$-a(\omega-\Omega) = (\kappa/ma)\,[\beta_0\, j - \tfrac{3}{2}\gamma_0(1+2\gamma_0^2)^{\frac{1}{2}}\,\phi_1\,\phi_2] \qquad (18a)$$

and

$$-b(\omega-\Omega) = (\kappa/mb)\,[\beta_0\, j + \tfrac{3}{2}\gamma_0(1+2\gamma_0^2)^{\frac{1}{2}}\,\phi_1\,\phi_2]. \qquad (18b)$$

These are the two relations between $(\omega-\Omega)$, β_0 and γ_0 referred to in the Introduction. It is possible to eliminate β_0 from these relations, and to obtain the relation

$$(b^2-a^2)\,(\omega-\Omega) = -(3\kappa/m)\,\gamma_0(1+2\gamma_0^2)^{\frac{1}{2}}\,\phi_1\,\phi_2. \qquad (19)$$

So $(\omega-\Omega)$ is directly related, as would be expected, to γ_0, the measure of the angular momentum transfer. If γ_0 is negative (as for the case of a transfer of westerly angular momentum towards $r = 0$, or towards the poles in the analogous case on a spherical earth), it follows that $\omega > \Omega$, and this is observed in the experiment. Using the results for the main motion of the fluid which are summarized in the Introduction to this paper, (19) can be used to give the expression

$$\frac{(b^2-a^2)\,\omega(\omega-\Omega)}{gh}\bigg/\frac{\Delta\rho}{\rho_0} = -\frac{3\gamma_0(1+\gamma_0^2)^{\frac{1}{2}}\,\phi_1\,\phi_2\,\epsilon_0}{2m}, \qquad (20)$$

and this is to be compared with equation (1), the experimental result obtained by Hide (1958). Since the ratio $(\omega-\Omega)/\Omega$ is small, it is permissible in equation (20) to put $\omega/\Omega = 1$, and then the two equations are identical as long as

$$3\gamma_0(1+2\gamma_0^2)^{\frac{1}{2}}\,\phi_1\,\phi_2\,\epsilon_0/2m = 5{\cdot}8\times 10^{-2}.$$

This can be regarded as an expression defining the quantity γ_0, and if γ_0^2 is very small compared with unity, the equation becomes

$$\gamma_0 \doteqdot mR_0^2/400kK^2\epsilon_0, \qquad (21)$$

which is negative, since $1/\epsilon_0 = -gh\Delta\rho/2\omega\rho_0\,\kappa$ is negative.

Note that, using equations (18a) and (18b), it is also possible to evaluate the constant β_0; for,

$$\begin{aligned} 2\kappa\beta_0\, j/m &= -(a^2+b^2)\,(\omega-\Omega) \\ &= \frac{(a^2+b^2)}{(b^2-a^2)}\frac{gh}{\omega}\frac{\Delta\rho}{\rho_0}\times 5{\cdot}8\times 10^{-2} = -\frac{(a^2+b^2)}{(b^2-a^2)}\frac{2\kappa}{\epsilon_0}\times 5{\cdot}8\times 10^{-2}. \end{aligned}$$

Hence,

$$\beta_0 = -5{\cdot}8\times 10^{-2}\,\frac{m}{j\epsilon_0}\frac{(a^2+b^2)}{(b^2-a^2)}. \qquad (22)$$

Using Davies's equations, summarized earlier in this paper,

$$m/j\epsilon_0 = R_0[1+O(k^2,\gamma_0^2)].$$

So, for sufficiently small k, γ_0, (22) becomes

$$\beta_0 \doteqdot -6\times 10^{-2}\frac{(a^2+b^2)}{(b^2-a^2)}\log_e(b/a) \doteqdot -12\times 10^{-2}\frac{(a^2+b^2)}{(b^2-a^2)}\frac{(b-a)}{(b+a)} \doteqdot -0{\cdot}1.$$

There are as yet no experimental results available to check the accuracy of these values of β_0 and γ_0.

Using equation (2) and the first half of the boundary condition (6a), we find that

$$u = (mF/\mu a)\{\cos[m\theta-\mu(r-a)+\alpha]-\cos(m\theta+\alpha)$$
$$+e^{-\sqrt{3}\mu(r-a)/2}\cos[m\theta+\tfrac{1}{2}\mu(r-a)+\alpha-\tfrac{1}{3}\pi]-\cos(m\theta+\alpha-\tfrac{1}{3}\pi)\},$$

and similarly, near the outer boundary,

$$u^* = (mF^*/\mu^* b)\{\cos[m\theta-\mu^*(r-b)+\alpha^*]-\cos(m\theta+\alpha^*)$$
$$+e^{\sqrt{3}\mu^*(r-b)/2}\cos[m\theta+\tfrac{1}{2}\mu^*(r-b)+\alpha^*+\tfrac{1}{3}\pi]-\cos(m\theta+\alpha^*+\tfrac{1}{3}\pi)\}.$$

The profiles of u and u^* when $r-a = 2\pi/\mu$, $r-b = -2\pi/\mu^*$ are indicated in the schematic sketch in figure 3: here the circular boundaries are represented by straight lines and the width of the boundary layers is considerably exaggerated.

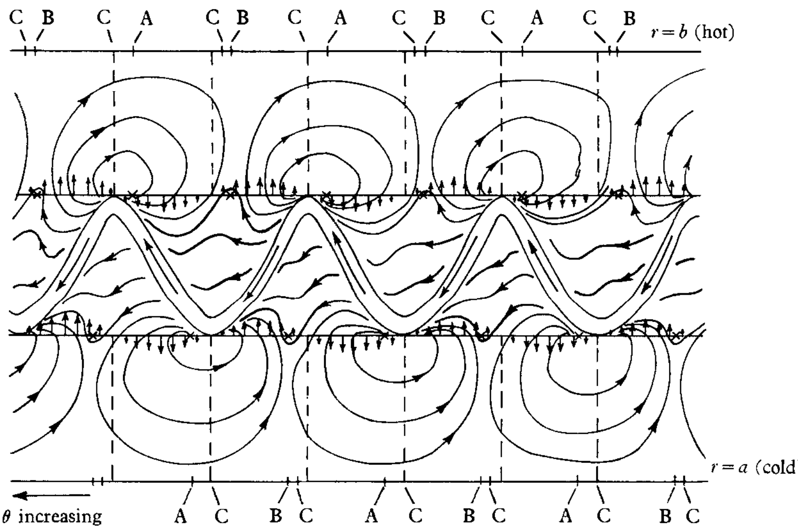

FIGURE 3. Schematic diagram of the radial velocity at the 'edge' of the boundary layer and of the streamlines in the boundary layer. A is a point of maximum v at the 'edge' of the boundary layer. B is a point of minimum v at the 'edge' of the boundary layer. C is a point of zero u at the 'edge' of the boundary layer.

Allowance has been made for the fact that, for values of θ where the transverse velocity, v, in the inviscid flow is smallest, the exponential term in the expressions for u and v in this paper is relatively more important in spite of its small absolute magnitude: in these regions, therefore, the boundary layer may be wider than in the regions where the jet stream approaches the wall. This effect cannot easily be incorporated into the equations, but it can be shown that the qualitative effect is to displace one position of zero u as indicated in the diagram. The general directions of the streamlines are sketched in the figure, and are in reasonably good agreement with experimental observation.

It is interesting to observe that the position of maximum transverse velocity at the 'edge' of the boundary layer defined in (14) is slightly upstream (relative to the jet stream) of the point where the jet stream meets the inner, cold boundary (given by $u = 0$), but slightly downstream of the point where the stream meets the outer, hot boundary. This implies that the fluid is accelerated at the inner boundary and decelerated at the outer boundary. No explanation is at present offered of this result, which is the opposite of that which would be expected at first sight.

4. The unsteady case : solution periodic in time

Any disturbance generated in the motion must be transmitted to the fluid through the boundary layer. If there exists, therefore, a natural frequency with which periodic oscillations can persist in the boundary layer, it seems likely that such oscillations will sometimes occur, and that they may be transmitted to the main body of the fluid. We look, therefore, for a solution which varies periodically with a frequency σ radians per second. If this exists, there will be a solution of equation (5a) in the form

$$\mathrm{p} = p_0(r,z) + p_1(r,z)\,e^{im\theta} + p_2(r,z)\,e^{-im\theta} + p_3, \tag{23}$$

where p_0, p_1 and p_2 are as in § 3, and

$$\begin{aligned} p_3 = e^{i\sigma t}[q_0(r,z) + q_1(r,z)\,e^{im\theta} + q_2(r,z)\,e^{-im\theta}] \\ + e^{-i\sigma t}[Q_0(r,z) + Q_1(r,z)\,e^{im\theta} + Q_2(r,z)\,e^{-im\theta}]. \end{aligned} \tag{24}$$

Substituting this in equation (5a), we find that the functions $q_0, q_1, q_2, Q_0, Q_1, Q_2$ satisfy the equations

$$i\sigma\frac{\partial q_0}{\partial r} = \nu\frac{\partial^3 q_0}{\partial r^3}, \quad -i\sigma\frac{\partial Q_0}{\partial r} = \nu\frac{\partial^3 Q_0}{\partial r^3}, \tag{25}$$

$$\left.\begin{aligned} i\sigma\frac{\partial q_1}{\partial r} = -\frac{2i\omega m}{a}q_1 + \nu\frac{\partial^3 q_1}{\partial r^3}, \quad -i\sigma\frac{\partial Q_1}{\partial r} = -\frac{2i\omega m}{a}Q_1 + \nu\frac{\partial^3 Q_1}{\partial r^3}, \\ i\sigma\frac{\partial q_2}{\partial r} = \frac{2i\omega m}{a}q_2 + \nu\frac{\partial^3 q_2}{\partial r^3}, \quad -i\sigma\frac{\partial Q_2}{\partial r} = \frac{2i\omega m}{a}Q_2 + \nu\frac{\partial^3 Q_2}{\partial r^3}. \end{aligned}\right\} \tag{26}$$

Equations (25) have solutions of the form

$$\left.\begin{aligned} q_0 = e^{(1+i)\,r\sqrt{(\sigma/2\nu)}}, \quad e^{-(1+i)\,r\sqrt{(\sigma/2\nu)}}, \quad \text{and a constant,} \\ Q_0 = e^{(1-i)\,r\sqrt{(\sigma/2\nu)}}, \quad e^{-(1-i)\,r\sqrt{(\sigma/2\nu)}}, \quad \text{and a constant.} \end{aligned}\right\} \tag{27}$$

Also, equations (26) have solutions of the form

$$\left.\begin{aligned} q_1 = e^{k_1 r}, e^{k_2 r}, e^{k_3 r}; \quad q_2 = e^{-k_1 r}, e^{-k_2 r}, e^{-k_3 r}; \\ Q_1 = e^{-\bar{k}_1 r}, e^{-\bar{k}_2 r}, e^{-\bar{k}_3 r}; \quad Q_2 = e^{\bar{k}_1 r}, e^{\bar{k}_2 r}, e^{\bar{k}_3 r}; \end{aligned}\right\} \tag{28}$$

here k_1, k_2, k_3 are roots of the cubic equation

$$k^3 - (i\sigma/\nu)\,k - 2i\omega m/\nu a = 0,$$

and $\bar{k}_1, \bar{k}_2, \bar{k}_3$ are the complex conjugates of these quantities. It is convenient to write

$$(a/\upsilon) = 3\eta\mu^2, \ \mu = (2\omega m/\nu a)^{\frac{1}{3}}, \tag{29}$$

so that k satisfies the equation

$$k^3 - 3i\eta\mu^2 k - i\mu^3 = 0. \tag{30}$$

If we assume that the oscillation investigated here is connected with that of the vacillation cycle described in the Introduction, we can use the values given there for one particular case: in this case we find that $\mu \doteqdot 14\,\text{cm}^{-1}$ and $\eta \doteqdot 0{\cdot}04$ at the inner boundary, and the corresponding quantities at the outer boundary are $\mu^* \doteqdot 8{\cdot}5\,\text{cm}^{-1}$, $\eta^* \doteqdot 0{\cdot}1$. It seems reasonable, therefore, to find a solution for k as a power series in η, and it is easy to verify that, if $k_0^3 = i\mu^3$, then

$$k = k_0\{1 + (k_0/\mu)\,\eta + O(\eta^3)\}.$$

So, neglecting terms of order η^3 compared with unity, the roots of equation (30) are

$$k_1 = -\mu\eta - i\mu, \quad k_2 = \tfrac{1}{2}\mu(\sqrt{3}+\eta) + \tfrac{1}{2}i\mu(1+\eta\sqrt{3}), \quad k_3 = \tfrac{1}{2}\mu(-\sqrt{3}+\eta) + i\mu(1-\eta\sqrt{3}).$$

Using these values in (28), and using also equations (27) and (24), it can be seen that, if

$$v = \frac{1}{2\omega\rho_0}\frac{\partial p_3}{\partial r},$$

then

$$\begin{aligned} v_3 = {} & F_0\, e^{(r-a)\sqrt{(\sigma/2\nu)}} \cos\{\sigma t + (r-a)\sqrt{(\sigma/2\nu)} + \alpha_{00}\} \\ & + G_0\, e^{-(r-a)\sqrt{(\sigma/2\nu)}} \cos\{\sigma t - (r-a)\sqrt{(\sigma/2\nu)} + \beta_{00}\} \\ & + F_1\, e^{-\mu\eta(r-a)} \cos\{\sigma t + m\theta - \mu(r-a) + \alpha_1\} \\ & + G_1\, e^{\mu\eta(r-a)} \cos\{\sigma t - m\theta + \mu(r-a) + \beta_1\} \\ & + F_2\, e^{\mu(\sqrt{3}+\eta)(r-a)/2} \cos\{\sigma t + m\theta + \tfrac{1}{2}(1+\eta\sqrt{3})\,\mu(r-a) + \alpha_2\} \\ & + G_2\, e^{-\mu(\sqrt{3}+\eta)(r-a)/2} \cos\{\sigma t - m\theta - \tfrac{1}{2}(1+\eta\sqrt{3})\,\mu(r-a) + \beta_2\} \\ & + F_3\, e^{-\mu(\sqrt{3}-\eta)(r-a)/2} \cos\{\sigma t + m\theta + \tfrac{1}{2}(1-\eta\sqrt{3})\,\mu(r-a) + \alpha_3\} \\ & + G_3\, e^{\mu(\sqrt{3}-\eta)(r-a)/2} \cos\{\sigma t - m\theta - \tfrac{1}{2}(1-\eta\sqrt{3})\,\mu(r-a) + \beta_3\}, \end{aligned}$$

where $F_0, F_1, F_2, F_3, G_0, G_1, G_2, G_3, \alpha_{00}, \alpha_1, \alpha_2, \alpha_3, \beta_{00}, \beta_1, \beta_2$, and β_3 are all arbitrary functions of z, and μ and η are defined in equation (29). Applying the boundary condition (7a) to the velocity v, it follows that $v_3 \to 0$ as $r-a \to \infty$, and hence $F_0 = G_1 = F_2 = G_3 = 0$. Further, applying the condition (6a), it follows that $v_3 = 0$ when $r = a$, and so

$$G_0 = 0, \quad F_1 + F_3 = 0, \quad \alpha_1 = \alpha_3, \quad G_2 = 0.$$

The time-dependent part of the velocity is therefore given by

$$\begin{aligned} v_3 = {} & F_1\, e^{-\mu\eta(r-a)} \cos\{\sigma t + m\theta - \mu(r-a) + \alpha_1\} \\ & - F_1\, e^{-\mu(\sqrt{3}-\eta)(r-a)/2} \cos\{\sigma t + m\theta + \tfrac{1}{2}(1-\eta\sqrt{3})\,\mu(r-a) + \alpha_1\}. \end{aligned} \tag{31a}$$

The corresponding expression for the time-dependent part of the velocity near the outer boundary is

$$\begin{aligned} v_3^* = {} & G_1^*\, e^{\mu^*\eta^*(r-b)} \cos\{\sigma^* t - m\theta + \mu^*(r-b) + \beta_1^*\} \\ & - G_1^*\, e^{\mu^*(\sqrt{3}-\eta^*)(r-b)/2} \cos\{\sigma^* t - m\theta - \tfrac{1}{2}(1-\eta^*\sqrt{3})\,\mu^*(r-b) + \beta_1^*\}, \end{aligned} \tag{31b}$$

where

$$\sigma^*/\nu = 3\eta^*\mu^{*2}, \quad \mu^* = (2\omega m/\nu b)^{\frac{1}{3}}. \tag{32}$$

Usually, of course, $\sigma^* = \sigma$, but at this stage the theory does not require this extra condition.

The boundary conditions (7a) and (7b) ensure that when $(r-a)$ or $(b-r)$ are sufficiently large, both v_3 and v_3^* vanish. Since, however, η is very much smaller than $\frac{1}{2}(\sqrt{3}-\eta)$, it follows that at the 'edge' of the steady boundary layer (defined in equation (14)), there is still a significant contribution from the first term in each of equations (31a) and (31b). Any disturbance in the transverse velocity v, then, at the outer boundary propagates as a forward travelling wave (relative to the direction of motion in the jet stream) which is sloping forward at least for large $(b-r)$. This is indicated in figure 4. Similarly, at the inner boundary, any disturbance will be propagated as a backward travelling wave with a backward slope. It seems likely that, as these waves travel round the cylinders and at the same time inwards towards the main body of the fluid (this is true at both boundaries),

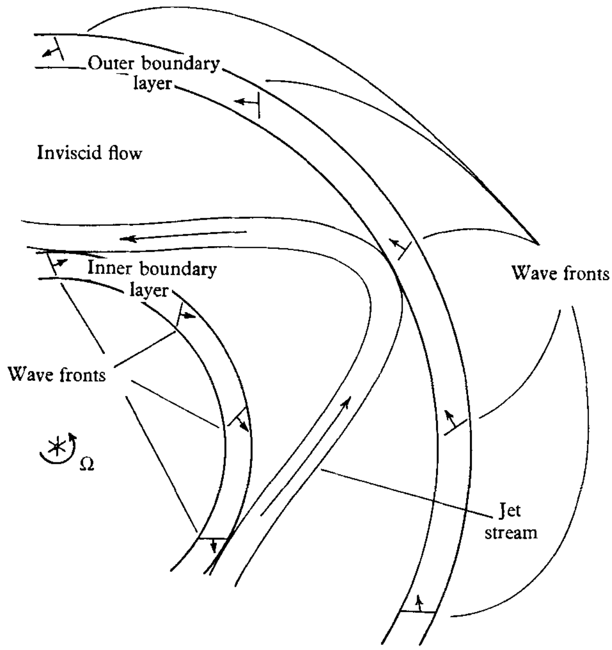

FIGURE 4. Wavefronts of disturbances in the boundary layers, moving into the fluid.

there will be a tendency for the lobes of the jet-stream pattern to tilt, and finally to roll up on themselves as shown in figure 5, and as observed in one cycle of the vacillation phenomenon. In fact, the present author has seen a cycle set up in the experimental work in Newcastle, in which a pattern very similar to that in figure 5 was periodically obtained, although the whole vacillation cycle did not in that case occur.

The slight opposite tilt which occurs during another phase of the vacillation cycle may in fact be present always, since the inviscid solution itself has such a backward tilt. This is evident, since

$$\Psi_0^* - \Psi_0 = -\frac{2\gamma_0}{(1+2\gamma_0^2)^{\frac{1}{2}}}\log_e\left(\frac{1-k}{1+k}\right),$$

and this is the difference in the angle θ between, say, the point where the jet stream touches the outer cylinder, and the point where it touches the inner cylinder. Since k is always positive, and in the present problem, γ_0 is negative, it follows that Ψ_0^* is less than Ψ_0—that is, there is a backward tilt to the whole pattern.

The whole vacillation cycle is not quite complete on this theory, as there is no obvious reason why the pattern should go through the symmetrical position when progressing from the rolled-up state to the backward tilt of the inviscid solution. But it is hardly to be expected that the present theory would account for this part of the cycle, since it is obviously unrealistic to treat the motion as basically that of Davies's solution.

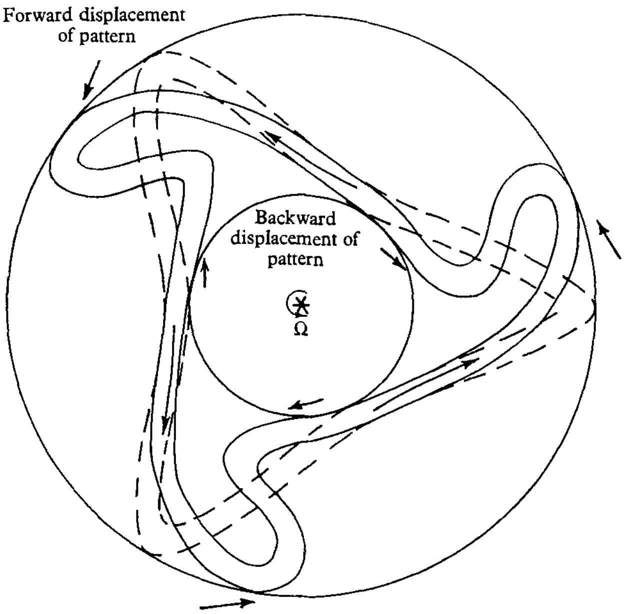

FIGURE 5. The tilting of the wave pattern at the boundaries during one phase of the cycle of an oscillating system.

For the same reason, it is hardly to be expected that the period, $2\pi/\sigma$, of this oscillation will be identical with that of the vacillation cycle; on the other hand, it is to be expected that the order of magnitude of the two periods will be the same. If we take $\sigma^* = \sigma$, which is certainly the case for vacillation, and assume that the maximum value of η (or η^*) for which the phenomenon can occur is approximately 0·1 (for larger values of η, the contribution of the time-dependent part of the transverse velocity is negligible outside the boundary layer), then the minimum possible value of the period of oscillation is

$$2\pi/\sigma = 20\pi/3\nu\mu^2 \quad \text{or} \quad 20\pi/3\nu\mu^{*2},$$

whichever is the greater. This suggests that the vacillation period is of order

$$2\pi/\sigma = (20\pi/3\nu)/(\nu b/2\omega m)^{\frac{2}{3}},$$

and this is about 30 sec in the particular case mentioned in the Introduction. It must be remembered, however, that the theory *assumes* that η is small in the first place, so that this result may be built into the system. At least the result shows that there is no inconsistency occurring.

If the tilt of the wave is very small, there will be a tendency for any disturbance to travel round and round the apparatus, remaining more or less within the boundary layer, instead of propagating into the fluid. A measure of the tilt at the inner and outer boundaries, respectively, are the quantities

$$\Phi = \mu a/m = (2\omega a^2/\nu m^2)^{\frac{1}{3}} \quad \text{and} \quad \Phi^* = \mu^* b/m = (2\omega b^2/\nu m^2)^{\frac{1}{3}}. \qquad (33)$$

If these parameters are sufficiently small, no significant fluctuations will occur in the main body of the fluid, and vacillation will not occur. If, in any given experiment, ω is gradually increased from a small value, both Φ and Φ^* increase at first; but after a certain critical value, m suddenly increases and so Φ and Φ^* both decrease discontinuously. This occurs repeatedly until the maximum value of m has been reached; after this increases of ω imply increases of Φ and Φ^* always. Hence, once m has reached its maximum value, there is a high probability that Φ and Φ^* will increase beyond their critical values and vacillation will occur. It is evident that Φ^* is greater than Φ, and so disturbances near the outer boundary will be significant for smaller values of ω than those near the inner boundary, and this is a possible reason for the earlier occurrence of oscillations in that region.

5. The present state of the problem: general discussion

The theory of this paper is intended to be only a stepping-stone to a fuller understanding of the wave-régime. In this section an attempt is made to put it in perspective relative to the general problem, and for the sake of simplicity the discussion will be restricted to the case in which there is a net heat flow towards the centre of the apparatus.

As long as the motion is steady, the main body of the fluid moves under a balance of Coriolis and pressure forces, as shown by Davies (1959), and the heat flow is partly by conduction and partly by convection. This flow forms a jet stream which meanders regularly between the inner and the outer cylindrical boundaries, and viscous forces may apparently be neglected in the main body of the fluid since, in Davies's solution, the only singularities in the vorticity gradient occur as the boundaries, where slipping occurs. It is, therefore, necessary to presume the existence of boundary layers at both cylindrical boundaries and at the bottom of the fluid. (At the free surface, there will also be a boundary layer, but the drag caused by the air above the liquid is certainly small, and if the apparatus has a cover, will be zero.) It is reasonable to suppose that this would give a complete solution to the problem, although a uniqueness theorem is not at present available.

The present paper deals with the boundary layers at the cylindrical walls, and it has been found that some progress towards the complete solution has been made, but that a third relation, in addition to the equations (18), is needed in order to determine the constants of the problem. In fact, even more is required, since the constant γ_0 in these equations occurs originally when Davies assumes a certain form ($\gamma_0 \phi = d\Psi/dR$) for the relation between ϕ and Ψ. It is desirable, therefore, to obtain, from a discussion of the boundary layer, conditions which will determine the form of this relation uniquely. A further disadvantage of Davies's solution as it stands is that it assumes a basic linear distribution of temperature in the vertical: Sullivan (1960) has extended his work and considers a more general distribution, but for a complete solution it is obviously necessary to predict, rather than to specify, such a distribution, and so far this has not been possible.

Another approach to that described is suggested by the result found in this paper that the boundary layers are thickest in the regions between the points where the jet stream touches the boundaries. This implies that viscous forces are more important in the slow vortex regions than in the jet stream, and it may be worth looking for a solution of the viscous equations in these regions and attempting to match them to the solution obtained by Rogers (1959) for the jet stream itself.

The vertical component of velocity has been ignored throughout: in fact it can be shown to be zero in the first approximation in the inviscid solution. It seems probable that fairly large vertical velocities do exist, however, and it may be that they are in the boundary layers discussed here. This seems a possibility as a means of producing the linear temperature gradient, on which all other variations are superposed, which was used by Davies. Further, such a vertical motion in the neighbourhood of the walls would help to explain the various energy exchanges in the fluid. On the other hand, it may be that all the vertical motions occur in the slow vortex regions. Here, then, is another promising topic for investigation.

The stability of the main motion has been discussed by Davies in the same paper, and he predicts results which agree with experiment. He cannot, however, consider the effects of disturbances from outside (such as a momentary change in angular velocity of the apparatus, which can easily occur), since such disturbances must be transmitted through viscous forces. It has been shown in the present paper that for small values of Φ and Φ^* (defined in equations (33)) it is unlikely that such disturbances will seriously affect the stability of the steady motion in the main fluid: this corresponds to comparatively small ω, and m less than its maximum value. When $m = m_{\max}$, however, an increase in ω leads first to a small oscillation of the wave pattern near the outer boundary, and then, if ω is increased still further, to a periodic 'rolling-up' of the pattern similar to that observed in the vacillation cycle. This is in complete agreement with experiment, and suggests that the main part of the flow in the vacillation cycle, if it may be taken to be inviscid, may be considered as a forced oscillation of the inviscid steady solution.

The present paper, then, is a single step in the process of building up a complete picture of the motion: the next step is probably to look for a solution for the boundary layer on the bottom of the fluid. Other approaches have, however, been shown to be possible, and it may be that these may be more amenable to analysis. Once a complete solution has been built up, using the various component parts which are gradually being developed at present, it should be possible to produce a unified theory which covers the whole problem; at least, such a result would be highly desirable. At present, however, we are a long way from this conclusion.

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